

THE PREPROJECTIVE PARTITION FOR HEREDITARY ARTIN ALGEBRAS

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ABSTRACT. The purpose of this paper is to study the preprojective partition of a hereditary artin algebra. For a hereditary algebra of finite representation type, we give some numerical invariants in terms of the length of chains of irreducible maps, also in terms of the length of the maximal indecomposable module, and the orientation of the quiver of the algebra. Similar results are given for algebras stably equivalent to hereditary artin algebras.

1. Introduction. Let Λ be an artin algebra (for instance a finite dimensional algebra over a field), and $\text{ind } \Lambda$ the full subcategory of the category of the finitely generated left Λ -modules, consisting of the indecomposable finitely generated modules. M. Auslander and S. O. Smalø proved in [8] that there is a unique collection of full subcategories $\{\underline{P}_i\}_{i \in \mathbb{N} \cup \infty}$ of $\text{ind } \Lambda$, where \mathbb{N} denotes the nonnegative integers, having the following properties:

- (a) If $A \in \underline{P}_i$ and $B \approx A$, then $B \in \underline{P}_i$ for all i .
- (b) $\underline{P}_i \cap \underline{P}_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i \in \mathbb{N} \cup \infty} \underline{P}_i = \text{ind } \Lambda$.
- (c) \underline{P}_i has only a finite number of nonisomorphic objects for each $i < \infty$.
- (d) For each $i < \infty$, an indecomposable module A is in \underline{P}_i , if and only if every surjective morphism $f: B \rightarrow A$ is a splittable surjection whenever every indecomposable summand of B is in $\bigcup_{k \geq i} \underline{P}_k$.

This uniquely determined collection $\{\underline{P}_i\}_{i \in \mathbb{N} \cup \infty}$ of subcategories of $\text{ind } \Lambda$ is called the preprojective partition of $\text{ind } \Lambda$ or the preprojective partition for Λ . Each \underline{P}_i ($i < \infty$) is called the i th preprojective class.

It follows immediately from the definition that \underline{P}_0 consists of the indecomposable projective Λ -modules. The dual notion is that of the preinjective partition which is denoted by $\{\underline{I}_i\}_{i \in \mathbb{N} \cup \infty}$ and again the \underline{I}_i 's ($i < \infty$) are called the preinjective classes. Also \underline{I}_0 consists of the indecomposable injective Λ -modules.

We also recall that an indecomposable module M is preprojective if $M \in \underline{P}_i$ for some $i < \infty$, similarly an $N \in \text{ind } \Lambda$ is preinjective if $N \in \underline{I}_j$ for some $j < \infty$.

The purpose of this paper is to study the preprojective partition for the case Λ is a hereditary artin algebra. In this case we have the following algorithm due to G. Todorov [16].

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THEOREM 1.1. *Let Λ be a hereditary artin algebra. Then an indecomposable nonprojective Λ -module Y is in \underline{P}_j ($j < \infty$) if and only if both of the following conditions hold:*

- (a) *There exist an indecomposable $X_0 \in \underline{P}_{j-1}$ and an irreducible map $X_0 \rightarrow Y$.*
- (b) *For every irreducible map $f: X \rightarrow Y$ with $X \in \text{ind } \Lambda$, $x \in \underline{P}_{j-1} \cup \underline{P}_j$.*

Also, it was shown [6, 8] that if Λ is hereditary, the preprojective modules are the modules of the form $\text{Tr } D^i P$ with i a nonnegative integer and $P \in \underline{P}_0$, and that the preinjective modules are the modules of the form $D \text{Tr}^j I$ for j nonnegative integers and $I \in \underline{I}_0$. We recall that $D: \text{mod } \Lambda \rightarrow \text{mod } \Lambda^{\text{op}}$ denotes the ordinary duality for artin algebras and $\text{Tr} = \text{Ext}_\Lambda^1(_, \Lambda)$ is the transpose. We must also point out that Auslander-Smalø's notion of preprojective modules coincides in the case of hereditary artin algebras with the original definition given by Dlab and Ringel [12].

Let Λ be an arbitrary artin algebra of finite representation type. Then, there are finitely many preprojective and preinjective classes, so we may ask whether their numbers are equal. This is not true in general, there are counterexamples for the case Λ is given by a Brauer tree (Ch. Riedtmann [13]) also when Λ is a quotient of a hereditary artin algebra (B. Rohnes [14]). We shall prove that these numbers are equal if Λ is hereditary or stably equivalent to a hereditary artin algebra. It is also well known [12] that if Λ is a hereditary artin algebra of finite representation type, then there is a unique indecomposable of maximal length. We show that the length of this module equals the number of preprojective classes. The proof uses the fact that the number of preprojective classes does not depend on the orientation of Λ , as well as some results from the theory of root systems. An identical result holds for artin algebras stably equivalent to a hereditary artin algebra.

Let Λ be an artin algebra. Then, we can associate to Λ a nonzero two sided ideal [8], $\underline{a}(\Lambda) = \text{ann}\{\underline{P}_i \mid i \text{ minimal such that } \underline{P}_{i-1} \text{ contains an injective module}\}$.

If Λ is an indecomposable hereditary artin algebra of finite representation type, we shall construct a sequence of hereditary artin algebras of finite representation type in the following way:

$\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_n$ where Λ_n is semisimple and $\Lambda_j = \Lambda_{j-1}/\underline{a}(\Lambda_{j-1})$. If we denote by $\underline{a}_k = \text{ann } \underline{P}_k(\Lambda)$, we show that each \underline{a}_{i+j} is the preimage ideal in Λ of $\underline{a}(\Lambda_j)$. The ideals $\text{ann } \underline{P}_j$ (for each j) are completely determined as the trace ideals in Λ of certain specified projective Λ modules.

Throughout this paper, Λ will denote a hereditary artin algebra. We will assume the basic results about almost split sequences, irreducible morphisms as well as the connection between the hereditary artin algebras and the representations of quivers (k -species). The reader will find the necessary details in the papers of M. Auslander and I. Reiten [1-4] and Dlab and Ringel [12].

We also use the following fact due to M. Auslander and M. I. Platzteck [6]: If M and N are two nonisomorphic indecomposable preprojective modules over a hereditary artin algebra, then every nonzero homomorphism $f: M \rightarrow N$ is a (finite) sum of nonzero compositions of irreducible morphisms passing through indecomposable modules.

2. The special indexing of the simple modules. Applications. Let Λ be a hereditary artin algebra and \mathcal{S} a complete set of nonisomorphic simple Λ modules S_1, \dots, S_n . We recall that there is a partial order on \mathcal{S} defined in the following way: $S_i < S_j$ if and only if $\text{Hom}_\Lambda(P(S_i), P(S_j)) \neq 0$, where $P(S_i)$ and $P(S_j)$ are the projective covers of S_i and S_j respectively.

An *admissible indexing* of the simple Λ modules is an order preserving bijection $a: \mathcal{S} \rightarrow \{1, 2, \dots, n\}$.

We devote this section to defining a special admissible indexing of the simple Λ modules for a certain class of hereditary artin algebras. This will enable us to describe in more detail some properties of the preprojective partition and in particular to show that the number of preprojective classes equals the number of preinjective classes if Λ is of finite representation type. The way to define this special indexing is closely related to some general properties of partially ordered sets. We start with some definitions.

DEFINITION. (a) A partially ordered set L is *connected*, if for every $a, b \in L$, there exists a sequence $a = a_1, a_2, \dots, a_n = b$ of elements in L with the property that every two consecutive elements are comparable.

(b) Let L be a connected partially ordered set and $a, b \in L$. We can then consider a chain $a = a_1, a_2, \dots, a_n = b$ having the property that for each i , a_i is either a successor or a predecessor of a_{i+1} . This type of chain is called *saturated*.

(c) Let L be a connected partially ordered set. L is said to contain a set of type $\tilde{A}_{k,m}$ with $k \neq m$, if there is an $a \in L$ and a saturated chain of elements in L , $a = a_1, a_2, \dots, a_n = a$ with the following properties: If k is the number of inequalities of the form $a_i < a_{i+1}$ and if m is the number of inequalities of the form $a_{i+1} > a_i$ for $i = 1, \dots, n-1$, then $m \neq k$.

For the remainder of this section, all the partially ordered sets will be connected and will have the property that they do not contain any subset of the type $\tilde{A}_{k,m}$ with $k \neq m$.

Let L be such a partially ordered set and let $a, b \in L$. Consider a saturated chain $a = a_1, \dots, a_n = b$. Then, define

$\eta(a, b)$ = the number of inequalities of the type $a_i < a_{i+1}$ in this chain,

$l(a, b) = \eta(a, b) - \eta(b, a)$.

We note that, because of our assumption on L , $l(a, b)$ does not depend on the choice of the saturated chain linking a to b .

If L is finite, we define $K(L) = \max\{l(a, b) \mid a, b \in L\}$.

LEMMA 2.1. (a) $l(a, b) = -l(b, a)$ for every $a, b \in L$.

(b) $l(a, c) = l(a, b) + l(b, c)$ for every $a, b, c \in L$.

(c) $K(L) = K(L^{\text{op}})$ where L^{op} denotes the opposite partially ordered set whose elements are the elements of L and where the inequalities are reversed.

REMARKS. Let Λ be an indecomposable hereditary artin algebra, \mathcal{I} the set of the nonisomorphic indecomposable injectives and \mathcal{P} the set of the nonisomorphic projectives. Then \mathcal{I} is a partially ordered set if we define $I < J$ if and only if

$\text{Hom}_\Lambda(I, J) \neq 0$. Similarly \mathcal{P} is a partially ordered set if we define $P < Q$ if and only if $\text{Hom}_\Lambda(P, Q) \neq 0$.

Also, \mathcal{P} and \mathcal{Q} are connected.

LEMMA 2.2. *Let Λ be an indecomposable hereditary artin algebra. Then:*

(a) *The following are equivalent.*

(i) *\mathcal{P} does not contain any subset of the type $\tilde{A}_{k,m}$ with $k \neq m$.*

(ii) *\mathcal{Q} does not contain any subset of the type $\tilde{A}_{k,m}$ with $k \neq m$.*

(b) *If Λ satisfies either condition in (a), the $K(\mathcal{Q}) = K(\mathcal{P})$.*

PROOF. The proof follows immediately from the following observation. If S and T are two simple Λ modules, then there is an irreducible map $P(S) \rightarrow P(T)$ if and only if there is an irreducible map $I(S) \rightarrow I(T)$.

If Λ is a hereditary artin algebra, indecomposable and such that \mathcal{P} does not contain any subset of the type $\tilde{A}_{k,m}$ with $k \neq m$, we define $K(\Lambda) = K(\mathcal{P}) = K(\mathcal{Q})$.

EXAMPLE. Since if Λ is of finite representation type, \mathcal{P} corresponds to a Dynkin diagram (see Dlab and Ringel [12]) it follows that in this case \mathcal{P} does not contain any subset of the type $\tilde{A}_{k,m}$ with $k \neq m$.

For the remainder of this paper, Λ will be a hereditary artin algebra having the property that \mathcal{P} does not contain any subset of the type $\tilde{A}_{k,m}$. Therefore, if Λ is indecomposable, $l(P, Q)$ is well defined for all indecomposable projectives P, Q and so is $K(\Lambda)$. Let \mathcal{S} be a complete set of nonisomorphic simple modules.

DEFINITION. We define layers $\mathcal{L}_1, \dots, \mathcal{L}_{K(\Lambda)+1}$ of simple Λ -modules as follows:

$$\begin{aligned}\mathcal{L}_1 &= \left\{ S \in \mathcal{S} \mid \max_{Q \in \mathcal{P}} l(P(S), Q) = K(\Lambda) \right\}, \\ \mathcal{L}_2 &= \left\{ S \in \mathcal{S} \mid \max_{Q \in \mathcal{P}} l(P(S), Q) = K(\Lambda) - 1 \right\}, \\ &\vdots \\ \mathcal{L}_{K(\Lambda)+1} &= \left\{ S \in \mathcal{S} \mid \max_{Q \in \mathcal{P}} l(P(S), Q) = 0 \right\}.\end{aligned}$$

Next, we arbitrarily order the simples in each layer, and for each simple S , we write $S = S_{i,j}$ where i means that $S \in \mathcal{L}_i$ and j means that in our arbitrary ordering of \mathcal{L}_i , S is in the j th place. Finally we order lexicographically the set of all simples $\mathcal{S} = \{S_{i,j}\}_{1 \leq i \leq K(\Lambda)+1; 1 \leq j \leq n_i}$.

PROPOSITION 2.3. (a) *Every simple $S \in \mathcal{L}_1$ is projective.*

(b) *Every simple $T \in \mathcal{L}_{K(\Lambda)+1}$ is injective.*

(c) *For all simples $S \in \mathcal{L}_i$ and $T \in \mathcal{L}_j$, $l(P(S), P(T)) = j - i$.*

PROOF. (a) and (c) are obvious. The second assertion follows from [16, Lemma 1.1.3] since $P(T)$ is a maximal projective, that is it cannot be properly embedded in an indecomposable projective.

DEFINITION. $\mathcal{S} = \{S_{i,j}\}_{1 \leq i \leq K(\Lambda)+1; j=1, \dots, n_i}$ ordered lexicographically is called a *special indexing* of the simple Λ -modules.

PROPOSITION 2.4. *Let Λ be an indecomposable hereditary artin algebra with the property that \mathcal{P} does not contain any subset of the type $\tilde{A}_{k,m}$ with $k \neq m$. Let $\mathcal{S} = \{S_{i,j}\}$ be a special indexing of the simple Λ -modules. Then \mathcal{S} is admissible.*

PROOF. We only need to show that $\text{Hom}_{\Lambda}(P(S), P(S')) \neq 0$ implies that $l(P(S), P(S')) \neq 0$. But $\text{Hom}_{\Lambda}(P(S), P(S')) \neq 0$ means that there is a chain of irreducible maps from $P(S)$ to $P(S')$. Therefore $l(P(S), P(S'))$ equals the length of that chain and we are done.

From now on, $\mathcal{S} = \{S_{i,j}\}$ will always denote a *special admissible indexing* of the simple Λ modules.

DEFINITION. Let Λ be a hereditary artin algebra. An indecomposable injective I is called *maximal* if there is no proper epimorphism $J \rightarrow I$ for an indecomposable injective J .

We recall the following:

LEMMA 2.5 [16]. *I is a maximal injective if and only if $\text{soc } I$ is projective.*

LEMMA 2.6. *Let Λ be a hereditary artin algebra of finite representation type and let \underline{P}_i be the first preprojective class containing an indecomposable injective I . Then I is a maximal injective.*

PROOF. This is clear, since otherwise there exist an injective indecomposable J and an epimorphism $J \rightarrow I$, so J must lie in a preceding preprojective class.

For a hereditary artin algebra of finite representation type, let I and J be two indecomposable injectives with $I \in \underline{P}_i$ and $J \in \underline{P}_j$. Define $m(I, J) = j - i$.

PROPOSITION 2.7. *Let Λ be an indecomposable hereditary artin algebra of finite representation type and let I and J be two indecomposable injective Λ -modules. Then $m(I, J) = l(I, J)$.*

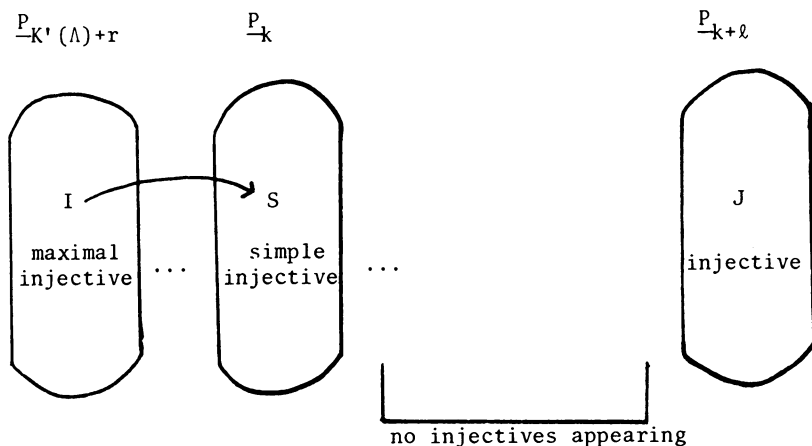
PROOF. If I, J, K are indecomposable injectives, it is clear that $m(I, J) = m(I, K) + m(K, J)$ so, since Λ is indecomposable it is enough to prove the equality for the case $\text{Hom}_{\Lambda}(I, J) \neq 0$. In this case, there is a chain of irreducible maps through indecomposable injectives: $I \rightarrow I_1 \rightarrow \cdots \rightarrow I_k = J$. Since Λ is hereditary, all the irreducible maps are epimorphisms and, since there are k irreducible maps in such a chain $l(I, J) = k$. Now, if $I \in \underline{P}_i$, $I_1 \in \underline{P}_{i+1}, \dots, I_k \in \underline{P}_{i+k}$ so $m(I, J) = k$, for $m(I, J) = m(I, I_1) + m(I_1, I_2) + \cdots + m(I_{k-1}, I_k)$ and $m(I_j, I_{j+1}) = 1$ for each j (using Theorem 1.1)

PROPOSITION 2.8. *Let Λ be an indecomposable hereditary artin algebra of finite representation type and let $\underline{P}_0, \dots, \underline{P}_{K'(\Lambda)}, \dots, \underline{P}_n$ be its preprojective partition, where $\underline{P}_{K'(\Lambda)}$ denotes the first preprojective class containing an injective.*

Then, for each $j \geq 0$, $\underline{P}_{K'(\Lambda)+j}$ contains an injective Λ -module.

PROOF. Assume that the claim is false, that is there exist $l > 1$ and $k > K'(\Lambda)$ such that the only injectives in \underline{P}_k are simple injectives and there are no injectives belonging to \underline{P}_j for $k+1 \leq j \leq k+l-1$.

We shall obtain a contradiction, and to prove it we shall use the illustration below.



Let I be a maximal injective mapping onto S . From our assumption on l , we have that J must also be a maximal injective. Consider the following sets:

$$\mathcal{Q} = \{\text{the injectives appearing in } \underline{P}_{K'(\Lambda)}, \dots, \underline{P}_k\},$$

$$\mathcal{B} = \{\text{the injectives appearing in } \underline{P}_{k+l}, \dots, \underline{P}_n\}.$$

So $\mathcal{Q} \cap \mathcal{B} = \emptyset$, and $\underline{I}_0 = \mathcal{Q} \cup \mathcal{B}$.

Since Λ is indecomposable, we can take the sequence $I = L_0, L_1, \dots, L_s = J$ with each L_i an indecomposable injective and such that for each $0 \leq j \leq s-1$ we have irreducible maps $L_j \rightarrow L_{j+1}$ or $L_{j+1} \rightarrow L_j$. Let j_0 be minimal with the property that L_{j_0} is not in \mathcal{Q} but $L_{j_0-1} \in \mathcal{Q}$. But now, this means that there is an irreducible map $L_{j_0-1} \rightarrow L_{j_0}$ and therefore $L_{j_0} \in \underline{P}_{k+l}$ giving a contradiction to the choice of l .

Let Λ be an indecomposable hereditary artin algebra of finite representation type. Then it is obvious from 1.1 that $K'(\Lambda)$ is the length of the shortest chain of irreducible morphisms linking a projective module to an injective one.

Using this observation we get:

THEOREM 2.9. *Let Λ be an indecomposable hereditary artin algebra of finite representation type and let $p(\Lambda)$ and $i(\Lambda)$ denote the number of preprojective and preinjective classes. Then $p(\Lambda) = i(\Lambda) = K(\Lambda) + K'(\Lambda) + 1$.*

PROOF. We show only that $p(\Lambda) = K(\Lambda) + K'(\Lambda) + 1$. The other equality follows by duality. Using the previous result, we see that $p(\Lambda) = K'(\Lambda) +$ the number of preprojective classes containing injectives. This latter is equal to $1 + \max m(I, J)$ and since $m(I, J) = l(I, J)$ for every two indecomposable injectives (see 2.7), it follows that $p(\Lambda) = K'(\Lambda) + K(\mathcal{J}) + 1 = K'(\Lambda) + K(\Lambda) + 1$.

THEOREM 2.10 (THE LOCATION OF THE INJECTIVES). *Let Λ be an indecomposable hereditary artin algebra of finite representation type and $\mathcal{S} = \{S_{i,j}\}$ be a special admissible indexing of the simple Λ -modules. Then, for each $0 \leq l \leq K(\Lambda)$, the injectives appearing in $\underline{P}_{K'(\Lambda)+l}$ are precisely the injective envelopes of $S_{l+1,j}$ for $1 \leq j \leq n_{l+1}$.*

PROOF. Let us first find the injectives in $\underline{P}_{K'(\Lambda)}$. $I \in \underline{P}_{K'(\Lambda)}$ if and only if $m(I, J) \geq 0$ for every injective J . This is equivalent to $l(I, J) \geq 0$ for every injective J , and furthermore to $l(P(\text{soc } I), Q) \geq 0$ for every indecomposable projective Q . The latter condition implies that $\text{soc } I = S_{1,j}$ for a j , that is $\text{soc } I \in \underline{\mathcal{C}}_1$.

The rest of the proof follows a similar pattern.

3. Trace ideals, annihilators and composition factors. We shall use the results obtained in the previous section to define inductively a sequence of two sided ideals in Λ ; $\underline{b}_1 \subset \underline{b}_2 \subset \underline{b}_3 \subset \cdots \subset \underline{b}_{K(\Lambda)}$ with the property that for each i , Λ/\underline{b}_i is hereditary. Then, if Λ is of finite representation type we show that $\underline{b}_i = \text{ann } \underline{P}_{K'(\Lambda)+i}$ for each i . Finally, we also describe which simples appear as composition factors for the modules in each preprojective class.

LEMMA 3.1. *Let Λ be an arbitrary artin algebra and S a simple Λ -module. If there exists a preprojective class \underline{P}_i ($i < \infty$) with the property that no modules in \underline{P}_i have S as composition factor, then the injective envelope of S is preprojective and lies in a \underline{P}_j ($j < i$).*

PROOF. Under our assumption, $I(S)$, the injective envelope of S , cannot be a quotient of a finite sum of modules in \underline{P}_i , so it must appear in a \underline{P}_j , $j < i$.

We have the following corollaries:

COROLLARY 3.2. *Let Λ be an indecomposable hereditary artin algebra of finite representation type and S a simple Λ -module that is not a composition factor for any module in a \underline{P}_i . Then, for each $l > i$, S does not appear as a composition factor for any module in \underline{P}_l .*

COROLLARY 3.3. *Let Λ be an indecomposable hereditary artin algebra of finite representation type. Then, for each $i \leq K'(\Lambda)$ and simple module S , S appears as a composition factor for some module in \underline{P}_i .*

PROOF. Since $\underline{P}_{K'(\Lambda)}$ is the first preprojective class containing an injective Λ -module, the corollary follows immediately from 3.1.

COROLLARY 3.4. *Let Λ be an indecomposable hereditary artin algebra of finite representation type. Let S be a simple Λ -module. Then, the following are equivalent for a \underline{P}_i :*

- (a) S does not appear as a composition factor for any module in \underline{P}_i .
- (b) $I(S)$ is in a \underline{P}_j ($j < i$).

PROOF. (a) \Rightarrow (b) was shown already.

(b) \Rightarrow (a) Assume that $I(S)$ is in \underline{P}_j ($j < i$). If there is an M in \underline{P}_{j+1} such that S is a composition factor for M , then S is a composition factor for an indecomposable summand of $I(M)$. Let us call this summand J . So, for a k , S is a summand of $r^k J / r^{k+1} J$. This means that there is an epimorphism from J onto $I(S)$, that is $I(S)$ is a quotient of a module in a \underline{P}_l ($l > j$). We obtain a contradiction and the corollary is proved.

DEFINITION. Let S be a simple Λ -module. We say that S is a composition factor for a preprojective class \underline{P}_i , if there is a module in \underline{P}_i having S as a composition factor.

THEOREM 3.5. *Let Λ be an indecomposable hereditary artin algebra of finite representation type and let $\mathfrak{S} = \{S_{i,j}\}$ be a special admissible indexing of the simple Λ modules. Then:*

- (a) *Every simple Λ module is a composition factor for $\underline{P}_0, \underline{P}_1, \dots, \underline{P}_{K'(\Lambda)}$.*
- (b) *For each $1 \leq i \leq K(\Lambda)$, $S_{i,1}, \dots, S_{i,n_i}$ are composition factors for $\underline{P}_0, \dots, \underline{P}_{K'(\Lambda)+i-1}$ and are not composition factors for the remaining preprojective classes.*

PROOF. We recall that we have shown that $I(S_{i,j}) \in \underline{P}_{K'(\Lambda)+i-1}$ for each $1 \leq i \leq K(\Lambda)$, and $1 \leq j \leq n_i$. This fact and the previous result complete the proof.

COROLLARY 3.6. *Let Λ be a hereditary indecomposable artin algebra of finite representation type. Let $\{S_1, \dots, S_n\}$ be the set of distinct simples appearing as composition factors for a \underline{P}_i and $P(S_1), \dots, P(S_n)$ their projective covers. Consider $\{T_1, \dots, T_m\}$, a complete set of nonisomorphic simples which are composition factors for $P(S_1) \amalg P(S_2) \amalg \dots \amalg P(S_n)$. Then, $\{S_1, \dots, S_n\} = \{T_1, \dots, T_m\}$.*

PROOF. It is left to the reader.

Our next objective is to determine the annihilator ideals of the \underline{P}_i 's. First, we recall the following lemma [3]:

LEMMA 3.7. *Let Λ be a hereditary artin algebra and let S_1, \dots, S_n be simples in the socle of Λ . Then the two sided ideal $\underline{b} = \tau_{S_1 \amalg \dots \amalg S_n}(\Lambda)$ (the trace of $S_1 \amalg \dots \amalg S_n$ in Λ) has the property that Λ/\underline{b} is a projective right Λ -module, and therefore the following hold for Λ/\underline{b} :*

- (a) Λ/\underline{b} is hereditary.
- (b) *Every indecomposable injective Λ/\underline{b} module is injective when viewed as a Λ module.*

It was shown by M. Auslander and I. Reiten [3] that if \underline{b} satisfies the hypothesis of the previous statement, and if $f: I \rightarrow J$ is an irreducible map between two indecomposable injective Λ/\underline{b} -modules, then f is irreducible as a Λ -homomorphism. This observation implies that the partially ordered set $\mathcal{J}(\Lambda/\underline{b})$ can be embedded in $\mathcal{J}(\Lambda)$, and by duality $\mathcal{P}(\Lambda/\underline{b})$ can be embedded in $\mathcal{P}(\Lambda)$ as a partially ordered set (by deleting the simple projectives S_1, \dots, S_n).

PROPOSITION 3.8. *Let Λ be an indecomposable hereditary artin algebra such that $\mathcal{P}(\Lambda)$ does not contain any subset of the type $\tilde{A}_{k,m}$ with $k \neq m$. Let $\mathfrak{S} = \{S_{i,j}\}$ be a special indexing of the simple Λ -modules. Consider the two sided ideals \underline{b}_i ,*

$$\underline{b}_i = \tau_{S_{1,1} \amalg \dots \amalg S_{1,n_1} \amalg \dots \amalg P(S_{i,1}) \amalg \dots \amalg P(S_{i,n_i})}(\Lambda)$$

for $i = 1, \dots, K(\Lambda)$. Then

- (a) Λ/\underline{b}_i are hereditary such that $\mathcal{P}(\Lambda/\underline{b}_i)$ do not contain any subsets of the type $\tilde{A}_{k,m}$ with $k \neq m$.

(b) $\Lambda/b_{K(\Lambda)}$ is semisimple.

(c) For every i , let \mathcal{S}_i be a special admissible indexing of the simple Λ/b_i -modules, that is $\mathcal{S}_i = \{T_{1,1}, \dots, T_{1,m_i}, \dots\}$. Let $\underline{b}'_i = \tau_{T_{1,1} \sqcup \dots \sqcup T_{1,m_i}}(\Lambda/b_i)$. Then, \underline{b}_{i+1} is the preimage ideal in Λ of \underline{b}'_i .

PROOF. (a) We use induction on i . If $i = 1$, we are done by the previous lemma and observation. Assume we have shown that Λ/b_i is hereditary. But then, since $P(S_{i+1,j})$ are simple projective Λ/b_i -modules for each $j = 1, \dots, n_{i+1}$, it follows that $\Lambda/b_{i+1} \approx (\Lambda/b_i)/(\underline{b}_{i+1}/b_i)$ which is again hereditary by 3.7.

(b) is trivial, and (c) is left to the reader.

DEFINITION. Let Λ be a hereditary artin algebra of finite representation type. For each i , we define the annihilator of \underline{P}_i , $\underline{a}_i = \bigcap \text{ann}\{M \mid M \in \underline{P}_i\}$.

It was shown by M. Auslander and S. Smalø [8] that $\underline{a}_0 = \underline{a}_1 = \dots = \underline{a}_{K'(\Lambda)} = 0$ and that $0 \neq \underline{a}_{K'(\Lambda)+1} \subset \underline{a}_{K'(\Lambda)+2} \subset \dots \subset \underline{a}_n$, where \underline{P}_n is the last preprojective class.

THEOREM 3.9. Let Λ be an indecomposable hereditary artin algebra of finite representation type and $\mathcal{S} = \{\mathcal{S}_{i,j}\}$ a special indexing of the simple Λ -modules. Let for $i = 1, \dots, K(\Lambda)$,

$$\underline{b}_i = \tau_{\mathcal{S}_{1,1} \sqcup \dots \sqcup \mathcal{S}_{1,n_1} \sqcup \dots \sqcup P(S_{i,1}) \sqcup \dots \sqcup P(S_{i,n_i})}(\Lambda).$$

Then, for each $1 \leq i \leq K(\Lambda)$, we have

$$\underline{b}_i = \underline{a}_{K'(\Lambda)+i} = \text{ann } \underline{P}_{K'(\Lambda)+i}.$$

PROOF. We already know that there are $K'(\Lambda) + K(\Lambda) + 1$ preprojective classes, so our notation makes sense.

Let M be in $\underline{P}_{K'(\Lambda)+i}$, that is it does not contain any simple Λ -module of the form $S_{k,l}$ ($1 \leq k \leq i$) as a composition factor. This means that $\text{Hom}_\Lambda(P(S_{k,l}), M) = 0$ using 3.6. Since for every Λ -module X , $\text{ann } X = \bigcap \text{Ker}\{f \mid f: \Lambda \rightarrow X\}$, we see that $\underline{b}_i \subset \text{ann } M$ for each M in $\underline{P}_{K'(\Lambda)+i}$. This shows that $\underline{b}_i \subset \underline{a}_{K'(\Lambda)+i}$. Now we show the other inclusion.

We know that

$$\begin{aligned} \underline{a}_{K'(\Lambda)+i} &\subset \bigcap \{ \text{ann } I \mid I \in \underline{P}_{K'(\Lambda)+j} \cap \mathcal{I} \text{ for each } j \geq i \} \\ &= \text{ann} \{ \bigsqcup I \mid I \text{ indecomposable injectives in } \underline{P}_{K'(\Lambda)+j}, j \geq i \}. \end{aligned}$$

But we have seen in 3.7 that these I 's are all the indecomposable injective Λ/b_i modules and using the first inclusion we get:

$$\begin{aligned} \underline{b}_i &\subset \underline{a}_{K'(\Lambda)+i} \subset \text{ann} \{ \bigsqcup I \mid I \text{ indecomposable injectives in } \underline{P}_{K'(\Lambda)+j}, j \geq i \} \\ &= \text{ann}_\Lambda(\Lambda/b_i) = \underline{b}_i. \end{aligned}$$

So, this ends the proof of the theorem.

4. Invariants given by the preprojective partition. We show in this section that if Λ is a hereditary artin algebra of finite representation type, then the number of preprojective classes equals the length of the unique indecomposable module of

maximal length. We use reduction to the case Λ is of radical square zero and the fact that the number of preprojective classes is invariant of the orientation of Λ . Using the same method, we show that $p(\Lambda) = L + 1$ where L is the length of the longest chain of irreducible maps with nonzero composition passing through indecomposable modules. First we recall some basic definitions and properties.

DEFINITION. Let Λ be a hereditary artin algebra, P_1, \dots, P_n the nonisomorphic indecomposable projective Λ -modules. Let $S = P_1$ be a simple projective noninjective Λ -module and $X = \text{Tr } DS \amalg P_2 \amalg \dots \amalg P_n$. Let $\Gamma = \text{End}_\Lambda(X)^{\text{op}}$. Then, the functor $F: \text{mod } \Lambda \rightarrow \text{mod } \Gamma$ defined by $F = \text{Hom}_\Lambda(X, _)$ is called *the left partial Coxeter functor* associated to S . These functors have been first introduced by I. N. Bernstein, I. M. Gelfand and V. A. Ponomarev [10] and were also studied by V. Dlab and C. M. Ringel in [12], S. Brenner and M. C. Butler [11]. Here, we shall use the above description [9].

It was shown by M. Auslander, M. I. Platzteck and I. Reiten in [9] that Γ is a finite representation type if and only if Λ is, that Γ is also hereditary, and that if $T = \text{Ext}'_\Lambda(X, S)$, then T is a simple injective Γ -module and $\text{Tr } DF(I(S)) = T$. In this context, we recall the following:

PROPOSITION 4.1 [9]. (a) *Let $\text{mod}_S \Lambda$ ($\text{mod}_T \Gamma$) denote the full subcategories of $\text{mod } \Lambda$ ($\text{mod } \Gamma$) consisting of the modules without direct summands isomorphic to S (T respectively). Then, the left partial Coxeter functor F induces an equivalence of categories*

$$F: \text{mod}_S \Lambda \rightarrow \text{mod}_T \Gamma.$$

(b) *Let $I(S)$ be the injective envelope of S . Then, the injective Γ -modules are of the form $\{F(I) \mid I \neq I(S)\} \cup T$, where $T = \text{Ext}'_\Lambda(X, S)$.*

The principal tool of this section is the following theorem:

THEOREM 4.2. *Let Λ be an indecomposable hereditary artin algebra of finite representation type and let S be a simple projective Λ -module. Let $F: \text{mod } \Lambda \rightarrow \text{mod } \Gamma$ be the left partial Coxeter functor determined by S . Then, $i(\Lambda) = i(\Gamma)$ where $i(\Lambda)$ and $i(\Gamma)$ are the numbers of the preinjective classes for Λ and Γ respectively.*

The proof of the theorem follows from the following proposition whose proof will be given later.

PROPOSITION 4.3. *Let Λ be an indecomposable hereditary artin algebra of finite representation type and $\mathcal{S} = \{S_{i,j}\}$ be a special admissible indexing for the simple Λ -modules. Let $p(\Lambda)$ (and $i(\Lambda)$) denote the numbers of preprojective (preinjective) classes for Λ . Let $P = P(S_{K(\Lambda)+1,j})$ for some $1 \leq j \leq n_{K(\Lambda)+1}$ and $I = I(S_{1,j})$ for some $1 \leq j \leq n_1$. Then*

(a) *Every chain of irreducible morphisms $P \rightarrow \dots \rightarrow P/rP$ has exactly $p(\Lambda) - 1$ irreducible morphisms.*

(b) *Every chain of irreducible morphisms $\text{soc } I \rightarrow \dots \rightarrow I$ has exactly $i(\Lambda) - 1$ irreducible morphisms.*

PROOF OF THEOREM 4.2. Let $\underline{I}_0(\Lambda), \dots, \underline{I}_n(\Lambda)$ be the preinjective classes for Λ . We know that $\underline{I}_n(\Lambda)$ contains only simple projective Λ -modules, and let $M \in \underline{I}_n(\Lambda)$. So, according to the proposition and the dual of 2.10, the shortest chain of irreducible morphisms linking M to an injective can be considered to be:

(*) $M \rightarrow M_{n-1} \rightarrow \dots \rightarrow M_1 \rightarrow I(M)$ with $M_i \in \underline{I}_i(\Lambda)$, and none of the M_i 's injective.

(a) *Case $M = S$.* Using the basic properties of the left partial Coxeter functor F , we have a chain of irreducible maps

$$F(M_{n-1}) \rightarrow F(M_{n-2}) \rightarrow \dots \rightarrow F(M_1) \rightarrow F(I(S)) \rightarrow F(J)$$

where J is a summand of $I(S)/S$. Therefore, in $\text{mod } \Gamma$ we have a chain of n irreducible maps linking $F(M_{n-1})$ to an injective Γ -module ($F(J)$), and this is the shortest such chain. Since $F(M_{n-1})$ is a projective Γ -module, and since $i(\Gamma) - 1$ is the supremum of the lengths of the shortest chains of irreducible maps starting from projective Γ -modules and ending at injective Γ -modules (this follows from Theorem 1.1), it follows that $i(\Lambda) \leq i(\Gamma)$.

(b) *Case $M \neq S$.* We get $F(M) \rightarrow F(M_{n-1}) \rightarrow \dots \rightarrow F(I(M))$ which is injective as a Γ -module, and this is again the shortest chain of irreducible maps linking $F(M)$ to an injective Γ -module. So again $i(\Lambda) \leq i(\Gamma)$. Therefore, $i(\Lambda) \leq i(\Gamma)$ in both cases. Next, we know by [9] that since Λ is of finite representation type, there is a sequence of hereditary artin algebras $\Lambda = \Lambda_0, \Lambda_1 = \Gamma, \dots, \Lambda_m = \Lambda$ and left partial Coxeter functors $F_i: \text{mod } \Lambda_{i-1} \rightarrow \text{mod } \Lambda_i$. Using the previous argument, we get

$$i(\Lambda) \leq i(\Gamma) = i(\Lambda_1) \leq i(\Lambda_2) \leq \dots \leq i(\Lambda_m) = i(\Lambda)$$

so that $i(\Lambda) = i(\Gamma)$.

It remains to prove the proposition. This will need a few steps whose proofs are immediate and are left to the reader.

LEMMA 4.4. *Let Λ be a hereditary artin algebra and $N \in \underline{P}_{i+1}$ ($i < \infty$). Then $D \text{Tr } N \in \underline{P}_{i-1} \cup \underline{P}_i$.*

PROOF. Follows from Theorem 1.1.

DEFINITION. Let Λ be a hereditary artin algebra and let $M \in \underline{P}_i$ ($i < \infty$). M is called maximal in \underline{P}_i , if there are no irreducible maps inside \underline{P}_i from M .

For instance, the maximal objects of \underline{P}_0 are the maximal projectives; also if I is injective in \underline{P}_i ($i < \infty$), then I is maximal since every irreducible map from I is an epimorphism.

It was shown for an arbitrary artin algebra by M. Auslander and S. Smalø [8] that \underline{P}_1 consists of the indecomposable modules of the form $\text{Tr } DM$ where $M \mid \underline{r}$, \underline{r} being the radical of Λ . For a hereditary artin algebra, this means that \underline{P}_1 consists of the modules of the form $\text{Tr } DP$ with P indecomposable nonmaximal projective. We have the following generalization:

LEMMA 4.5. *Let Λ be a hereditary artin algebra. Then, for each $1 < i < \infty$, $\underline{P}_{i+1} = \{\text{Tr } DX \mid X \text{ noninjective, maximal in } \underline{P}_{i-1}\} \cup \{\text{Tr } DY \mid Y \text{ nonmaximal in } \underline{P}_i\}$.*

THE PROOF OF PROPOSITION 4.3. We prove only (a) since (b) follows by duality. We recall that one of the assumptions of the proposition is that Λ is of finite representation type so that there are irreducible maps $P = P(S_{k(\Lambda)+1,j}) \rightarrow \cdots \rightarrow S = S_{k(\Lambda)+1,j}$. Since S is a simple injective, it is enough to show the following claim in order to prove the proposition.

Claim. Let Q be a summand of $\underline{r}P$. Then, for every $k \geq 1$, $\text{Tr } D^k P$ is in \underline{P}_{2k} and $\text{Tr } D^k Q$ is maximal in \underline{P}_{2k-1} (as long as $\text{Tr } D^{k-1} P$ and $\text{Tr } D^{k-1} Q$ are not injective).

PROOF OF THE CLAIM. Induction on k .

$k = 1$. Since $P \in \underline{P}_0$ is maximal, $\text{Tr } DP \in \underline{P}_2$ by 4.5 and $\text{Tr } DQ \in \underline{P}_1$. If $\text{Tr } DQ$ is not maximal in \underline{P}_1 , then there is an irreducible map $\text{Tr } DQ \rightarrow \text{Tr } DR$ in \underline{P}_1 with $R \mid \underline{r}R'$ for some $R, R' \in \underline{P}_0$. Now, there is an irreducible map $Q \rightarrow R$ so we get $l(P, R') = l(P, Q) + l(Q, R) + l(R, R') = -1 + 1 + 1 = 1$ a contradiction to the fact that $P = P(S_{K(\Lambda)+1,j})$.

Assume that $\text{Tr } D^{k-1} P \in \underline{P}_{2k-2}$ noninjective and that $\text{Tr } D^{k-1} Q$ as in \underline{P}_{2k-3} for each $Q \mid \underline{r}P$ and is maximal there. So, by 4.5, $\text{Tr } D^k Q \in \underline{P}_{2k-1}$. Let us take the almost split sequence: $0 \rightarrow \text{Tr } D^{k-1} P \rightarrow E \rightarrow \text{Tr } D^k P \rightarrow 0$. Since every summand of E has the form $\text{Tr } D^k Q$ with $Q \mid \underline{r}P$, it follows that $\text{Tr } D^{k-1} P \in \underline{P}_{2k-2}$ and is maximal so that $\text{Tr } D^k P \in \underline{P}_{2k}$.

Next, we prove the proposition. We want to show that if we have a chain of irreducible morphisms $P \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n = S$, then $X_i \in \underline{P}_i$ for each i .

Since $X_1 = \text{Tr } DQ$ for $Q \mid \underline{r}P$, it follows that X_1 is maximal in \underline{P}_1 by the claim and so $X_2 \in \underline{P}_2$. Next we have irreducible maps

$$\text{Tr } D^j X_1 \rightarrow \text{Tr } D^j X_2 \rightarrow \text{Tr } D^{j+1} X_1$$

and by the claim it must follow that $\text{Tr } D^j X_2$ is maximal in each preprojective class it belongs to, for each j . So, by induction it follows that X_i is maximal in each preprojective class it belongs to, and now the proposition follows.

Now we go on to prove our assertion about the maximal module. First we construct the preprojective partition for a hereditary artin algebra with radical square zero. We need first two lemmas. Their proof are straightforward and will be left to the reader.

LEMMA 4.6. *Let Λ be a hereditary algebra with radical square zero. Then, for each $0 < i < \infty$, $\text{Hom}_\Lambda(X, Y) = 0$ for every nonisomorphic X, Y in \underline{P}_i .*

PROOF. Since there are only simple and maximal projectives, \underline{P}_1 consists of $\text{Tr } DS$ (for S simple projectives) and using also Lemma 4.5, we get that each module in \underline{P}_i is maximal in \underline{P}_i for every $i \geq 1$. Since by [4] every nonzero morphism $f: X \rightarrow Y$ is a sum of compositions of irreducible maps, the lemma follows.

If Λ is an indecomposable hereditary artin algebra of finite representation type with $r^2 = 0$, then $K(\Lambda) = 2$ and thus the last preprojective class \underline{P}_n contains all the simple injectives, and thus \underline{P}_{n-1} consists of the maximal injectives, that is of the injectives whose socles are projective.

LEMMA 4.7. *Let Λ be an indecomposable hereditary artin algebra of finite representation type with radical square zero, and let P, Q be maximal projectives. Let i be such that $\text{Tr } D^i P$ is injective. Then $\text{Tr } D^i Q$ is also injective and lies in the same preprojective class as $\text{Tr } D^i P$.*

Let Λ be a hereditary artin algebra of finite representation type. If Λ is indecomposable, it was shown by Dlab and Ringel [12] that there is an equivalence between $\text{mod } \Lambda$ and the category of representations of k -species corresponding to a Dynkin diagram. So we can talk about the Coxeter transformation associated to the hereditary artin algebra Λ . (See [12].)

LEMMA 4.8 (G. TODOROV [16, PROPOSITION 3.4.3]). *Let Λ be an indecomposable hereditary artin algebra of finite representation type. Then, there exist P, Q indecomposable projectives such that:*

$$\text{Tr } D^{\sigma_1} P = I(Q/rQ),$$

$\text{Tr } D^{\sigma_2} Q = I(P/rP)$ and $\sigma_1 + \sigma_2 + 2 = \tau$ where τ is the order of the Coxeter transformation.

Using the results of Dlab and Ringel and a basic result from the theory of root systems (namely that $1 + h(R) = h$ where h is the order of the Coxeter elements and $h(R)$ is the height of the highest root), we have

LEMMA 4.9. *Let Λ be a hereditary indecomposable artin algebra of finite representation type and let M be the module of maximal length in $\text{ind } \Lambda$. Let τ be the order of the Coxeter transformation. Then, $L(M) + 1 = \tau$ where $L(M)$ is the length of M .*

We can prove now our main result:

THEOREM 4.10. *Let Λ be an indecomposable hereditary artin algebra of finite representation type. Let M be the maximal indecomposable Λ -module. Then $L(M) = p(\Lambda)$ where $p(\Lambda)$ denotes the number of preprojective classes for Λ .*

PROOF. Since it is well known that the order of the Coxeter transformation, and thus the length of the maximal module is an invariant of the orientation of Λ , and so is $p(\Lambda)$ by 4.2 and 2.9, it is enough to prove the theorem for the case Λ has radical square zero. So, we know by 4.8 that we can find two indecomposable projectives P_1 and P_2 and two indecomposable injectives $I_1 = I(P_1/rP_1)$ and $I_2 = I(P_2/rP_2)$ such that $\text{Tr } D^{\sigma_1} P_1 = I_2$ and $\text{Tr } D^{\sigma_2} P_2 = I_1$. We have to distinguish between three cases:

- (a) Both P_1 and P_2 are simple projectives.
- (b) Both P_1 and P_2 are maximal projectives.
- (c) One of them is a simple projective, the other is a maximal projective.

(a) In this case both I_1 and I_2 must be maximal injectives. So by a trivial analogue to Lemma 4.7, $\sigma_1 = \sigma_2$ and they both lie in \underline{P}_{n-1} where $p(\Lambda) = n + 1$. But then $P_i \in \underline{P}_0$, $\text{Tr } DP_i \in \underline{P}_1$, $\text{Tr } D^2 P_i \in \underline{P}_2$, etc. for each $i = 1, 2$. By counting the preprojective classes we get $2\sigma_i + 1 = p(\Lambda)$ and since $\sigma_1 = \sigma_2$ we get $2\sigma_i + 2 = p(\Lambda) + 1 = \tau$ for each i or $p(\Lambda) = \tau - 1 = L(M)$.

(b) is similar to (a).

(c) Assume for example that P_1 is maximal projective and that P_2 is a simple projective. Then I_1 is a simple injective and I_2 is a maximal injective. If $p(\Lambda) = n + 1$, we have

$$\text{Tr } DP_2 \in \underline{P}_1, \text{Tr } D^2P_2 \in \underline{P}_3, \dots, \text{Tr } D^{\sigma_2}P_2 \in \underline{P}_n, \text{ and}$$

$$\text{Tr } DP_1 \in \underline{P}_2, \text{Tr } D^2P_1 \in \underline{P}_4, \dots, \text{Tr } D^{\sigma_1}P_1 \in \underline{P}_{n-1}.$$

Again, if we count the classes we get: $2\sigma_2 = n + 1$ and $2\sigma_1 = n - 1$ and $2(\sigma_2 + \sigma_1) = 2n$ or $\sigma_2 + \sigma_1 + 1 = n + 1 = p(\Lambda)$. On the other hand $\sigma_1 + \sigma_2 + 2 = \tau = L(M) + 1$ and we are done.

Using the correspondence with the Dynkin diagrams we have the following table:

Λ	A_n	B_n	C_n	D_n	E_6	E_7	E_8	G_2	F_4
$p(\Lambda)$	n	$2n - 1$	$2n - 1$	$2n - 3$	11	17	29	5	11

The following result was suggested by Idun Reiten.

THEOREM 4.11. *Let Λ be an indecomposable hereditary artin algebra of finite representation type and let $L(\Lambda)$ equal the length of the longest chain of irreducible morphisms through indecomposables, having nonzero composition. Then $L(\Lambda) + 1 = p(\Lambda)$.*

PROOF. Since $p(\Lambda)$ was shown (4.2 and 2.9) to be an invariant of the orientation of Λ , it is enough to show that $L(\Lambda)$ is also an invariant of the orientation and prove the equality for the case $r^2 = 0$, where it becomes trivial due to 4.6.

Let S be a simple projective Λ -module, F the left partial Coxeter functor determined by S and $\Gamma = \text{End}_{\Lambda}(X)^{\text{op}}$ where $X = \text{Tr } DS \amalg P_2 \amalg \dots \amalg P_n$ where S_1, P_2, \dots, P_n are the nonisomorphic indecomposable projective Λ -modules. Let $M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_m$ be the longest chain of irreducible maps in $\text{mod } \Lambda$ with nonzero composition and the M_i indecomposable. If $M_0 \not\approx S$, then $F(M_0) \rightarrow F(M_1) \rightarrow \dots \rightarrow F(M_m)$ is again a chain of irreducible maps (in $\text{mod } \Gamma$) and the composition is nonzero, so $L(\Lambda) \leq L(\Gamma)$. If $M_0 \approx S$, then $M_m = I(S)$. So, there is a J dividing $I(S)/S$ such that $\text{Hom}_{\Lambda}(M_1, J) \neq 0$ and M_1 is also projective since S was a simple projective. Therefore $M_1 \rightarrow \dots \rightarrow M_m \rightarrow J$ is a chain of irreducible maps with nonzero composition, also of length m . We are again in the first case, so $L(\Lambda) \leq L(\Gamma)$. Now, there exists a chain of hereditary artin algebras $\Lambda = \Lambda_0, \Lambda_1 = \Gamma, \dots, \Lambda_k = \Lambda$ and left partial Coxeter functors $F_i: \text{mod } \Lambda_{i-1} \rightarrow \text{mod } \Lambda_i$ for $i = 1, \dots, k$, and according to our argument, $L(\Lambda) \leq L(\Gamma) \leq L(\Lambda_2) \leq \dots \leq L(\Lambda_{k-1}) \leq L(\Lambda)$ so that $L(\Lambda) = L(\Gamma)$ and we are done.

5. Artin algebras stably equivalent to a hereditary artin algebra. Let Λ and Γ be two artin algebras. We recall that Λ, Γ are stably equivalent if the categories $\text{mod } \Lambda$ and $\text{mod } \Gamma$ are equivalent, where $\text{mod } \Lambda$ denotes the category $\text{mod } \Lambda$ modulo the morphisms factoring through a projective module. Using the equivalences $\text{Tr } D: \text{mod } \Lambda \rightarrow \text{mod } \Lambda$, where $\text{mod } \Lambda$ is $\text{mod } \Lambda$ modulo the morphisms factoring through an injective module, we see that if Λ and Γ are stably equivalent, then $\text{mod } \Lambda$ and $\text{mod } \Gamma$ are equivalent categories. Throughout this section Λ denotes an artin algebra stably equivalent to a hereditary artin algebra Γ . First we recall some basic results. For details, see [3, 5, 7, 8].

LEMMA 5.1. *Let M, N be indecomposable nonprojective Γ -modules and $f: M \rightarrow N$ an irreducible map. Then $F(f): F(M) \rightarrow F(N)$ can be lifted to an irreducible map in $\text{mod } \Lambda$ where $F: \text{mod } \Gamma \rightarrow \text{mod } \Lambda$ is an equivalence.*

LEMMA 5.2 [5]. *Let $f \in \text{Hom}_\Lambda(X, Y)$ such that $f \neq 0$. Then, the following are equivalent:*

- (a) *f is an epimorphism in $\text{mod } \Lambda$.*
- (b) *f is an epimorphism in $\text{mod } \Gamma$.*

LEMMA 5.3 [7]. (a) *If M is a preprojective Γ -module, then $F(M)$ is a preprojective Λ -module where $F: \text{mod } \Gamma \rightarrow \text{mod } \Lambda$ is an equivalence.*

(b) *If X is preprojective in $\text{mod } \Lambda$ and $X \rightarrow Y$ is irreducible with Y indecomposable, then Y is also preprojective.*

LEMMA 5.4. (a) $F(\underline{P}_i(\Gamma)) = \underline{P}_i(\Lambda)$ for each $i \geq 1$.

(b) *For each $X \in \underline{P}_i(\Lambda)$ and $X \rightarrow Y$ irreducible with $Y \in \text{ind } \Lambda$, $Y \in \underline{P}_0 \cup \underline{P}_i \cup \underline{P}_{i+1}$ for each i .*

PROOF. It is clear that (b) follows from (a) using Todorov's result (1.1).

(a) Induction on i . Assume $\underline{P}_1(\Gamma) = \{C_1, \dots, C_n\}$. First we must show that $\{F(C_1), \dots, F(C_n)\}$ is a covering for $\text{ind } \Lambda - \underline{P}_0(\Lambda)$, that is it contains $\underline{P}_1(\Lambda)$. Let X be an indecomposable nonprojective Λ -module, nonisomorphic to any of the $F(C_i)$, that is $G(X) \not\approx C_i$ for any $i = 1, \dots, n$, where G is the equivalence $G: \text{mod } \Lambda \xrightarrow{\sim} \text{mod } \Gamma$.

Therefore, in $\text{mod } \Gamma$, there is an epimorphism $f: \coprod m_i C_i \rightarrow G(X)$, and since Γ is hereditary, $f = \underline{f}$ is an epimorphism in $\text{mod } \Gamma$ so $F(\underline{f})$ is an epimorphism in $\text{mod } \Lambda$, and by 5.2, there is an epimorphism in $\text{mod } \Lambda$ from $\coprod m_i F(C_i)$ onto $X \approx \overline{FG}(X)$. This shows that $\underline{P}_1(\Gamma) \subset \{F(C_1), \dots, F(C_n)\}$. Assume $\{F(C_1), \dots, F(C_n)\}$ is not a minimal cover, that is there exists for example an epimorphism $f: \coprod_{i \neq 1} n_i F(C_i) \rightarrow F(C_1)$ in $\text{mod } \Lambda$. If $f \neq 0$ we are done by the Lemma 5.2. So, assume $f = 0$. So f factors through the projective cover of $F(C_1)$, and that means that one of the $F(C_i)$ would be projective which gives a contradiction.

The case $i > 1$ is identical.

COROLLARY 5.5. *Let Λ be an artin algebra stably equivalent to a hereditary artin algebra Γ , and assume Λ is of finite representation type. Let $p(\Lambda)$ and $i(\Lambda)$ be the numbers of preprojective and preinjective classes.*

Then $p(\Lambda) = i(\Lambda) = p(\Gamma)$.

PROOF. Using the previous result we get $p(\Lambda) = p(\Gamma)$. Its dual gives $i(\Lambda) = i(\Gamma)$. Since $p(\Gamma) = i(\Gamma)$, the corollary follows.

LEMMA 5.6 [8]. *Let Λ be an arbitrary artin algebra of finite representation type and let $p(\Lambda)$ be the number of preprojective classes for Λ . Let P be an indecomposable projective Λ module. Then $L(P) \leq p(\Lambda)$.*

Next we recall the relationship between the lengths of modules in $\text{mod } \Lambda$ and $\text{mod } \Gamma$ where Λ is an artin algebra stably equivalent to a hereditary artin algebra Γ .

Let $F: \text{mod } \Gamma \rightarrow \text{mod } \Lambda$ and $F': \overline{\text{mod } \Gamma} \rightarrow \overline{\text{mod } \Lambda}$ be the equivalences. The result is due to M. Auslander and I. Reiten [5].

LEMMA 5.7. (a) For each indecomposable nonprojective Γ -module M , $L(M) = L(F(M))$.

(b) For each indecomposable noninjective Γ -module M , $L(M) = L(F'(M))$.

(c) Assume that Γ is Nakayama. Let P be the indecomposable projective injective Γ -module i.e. the Γ -module of maximal length. Then

(i) Λ is Nakayama.

(ii) There is a projective injective Λ -module Q with the property that $L(P) = L(Q)$ and Q is the unique Λ -module of maximal length.

As a trivial consequence we get

COROLLARY 5.8. Let Λ be an artin algebra of finite representation type, stably equivalent to a hereditary artin algebra Γ . Let M be the unique indecomposable Γ module of maximal length. Then

(a) There is a unique indecomposable Λ -module X of maximal length $L(X) = L(M)$.

(b) Let X be the module of maximal length in $\text{mod } \Lambda$. Then $X = F(M)$ and $L(X) = p(\Lambda)$.

PROOF. We know that $L(M) = p(\Gamma)$ (4.10). (a) is a trivial consequence of 5.7.

(b) If M is nonprojective, then $L(F(M)) = L(M)$ and it is immediate that in $\text{mod } \Lambda$ there are no indecomposables of length at least $L(F(M))$ except $F(M)$. So $L(F(M)) = L(M) = p(\Gamma) = p(\Lambda)$ so we have to show that $F(M) = X$ and we will be done. If $L(F(M)) < L(X)$, then X must be projective, so by 5.6, $L(X) < p(\Lambda)$ which gives a contradiction. The cases where M is noninjective and projective-injective are treated similarly.

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